NUMERICAL ANALYSIS

Numerical analysis is the study of algorithms that use numerical calculations to approximately solve the complex problems involving complex equations, functions and calculations.

Analytical solution: Solving a problem using mathematical theorems, finding the general form of a problem. Value at specific point and condition can be found by applying problem specific values at the closed form.

Numerical solution: Solving a problem without finding a closed form solution. The actual problem is solved at limited interval using numerical algorithms.

Analytical solutions: It is not possible to solve all problems analytically. Some acceptable hypothesis are made at engineering problems so that it can be solved analytically.

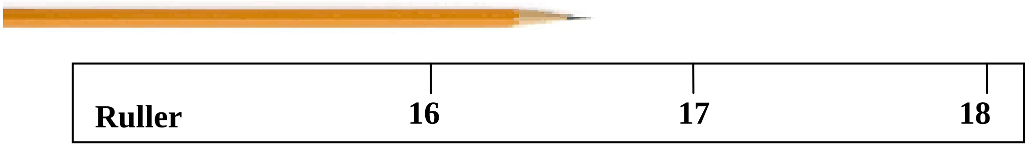
Numerical solutions: We can use numerical algorithms to find a required solution value needed for design. Unfortunately, numerical algorithms does not always converge to solution of the problem at required point.

1) DIGITS , APPROXIMATIONS, ERROR AND TAYLOR SERIES

1.1 Significant Figures and Decimal Figures.

Significant figures: The figures that can be used with confidence. We are sure that the figure has the close value to the actual result.

Example: What is the length of the pencil?

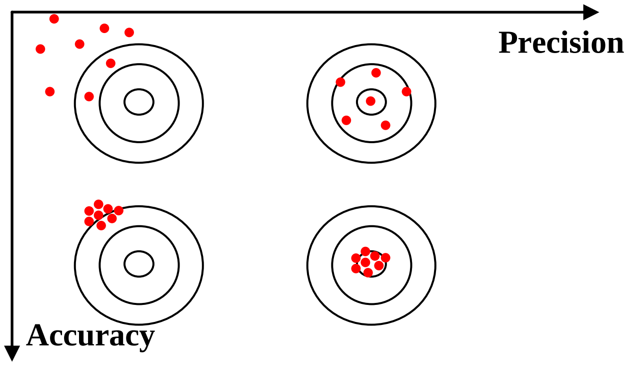


Example: State the number of significant digits of some number representations.

Decimal places: It is the number of digits used to represent a number. Sometimes it is required to designate the number of digits used after a point. These degits can be significant or not.

Example: Same numbers

1.2 Accuracy and Precision:



1.3 Approximation: Approximation is the use of a numerical method to find a number closer to the exact value of a problem at hand. The result is an approximate number.

Example: What is the value of π ?

Babylon : 3 (close)
Chinese : 22/7 = 3.14 (other digits are meaningless)
Indian (20th century) : 3.14156

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}}$$

1.4 Error Definitions:

Error is the measure of deviation from an actual value.

a) True Error (E_t) : Is the difference between the actual values and the approximate values.

$$E_t = True\ value - Approximation$$

Example: A company owner wants to hire a mechanical engineer. He will give responsibility of the firm manufacturing. For that, 2 candidates are given tasks to perform manufacturing jobs.

i) First candidate is given a job to manufacture a nail with 31.50 mm length. At the end of manufacturing, first candidate manufacture the nail with 32.08 mm. What is the true error of the manufacturing?

$$E_t = True\ value - Approximation$$
$$E_t = 31.50 - 32.08 = -0.58\ mm$$

ii) Second candidate is given a job to manufacture a nail with 1.20 mm width. At the end of manufacturing, first candidate manufacture the nail with 0.76 mm. What is the true error of the manufacturing?

$$E_t = True\ value - Approximation$$
$$E_t = 1.20 - 0.76 = 0.44\ mm$$

iii) As a firm manager, who do you hire? First candidate or second candidate? Discuss.

b) True Fractional (Relative) Error (ε_t) : It is the fraction of error with respect to true value.

$$\varepsilon_t = \frac{True\ value - Approximation}{True\ value}$$

Example: Let us calculate true fractional error this time.

i) $\varepsilon_t = \frac{31.50-32.08}{31.50} = -0.0184$

ii) $\varepsilon_t = \frac{1.20-0.76}{1.20} = 0.367$

c) True Percent (Relative) Error (%ε_t) : It is the representation of true fractional error in percent form.

$$\%\varepsilon_t = \frac{True\ value - Approximation}{True\ value} x100$$
$$\%\varepsilon_t = \varepsilon_t x100$$

Example: Let us calculate true fractional error this time.

i) $\varepsilon_t = \frac{31.50-32.08}{31.50} x100 = -1.84\%$

ii) $\varepsilon_t = \frac{1.20-0.76}{1.20} = 36.7\%$

***** In numerical analysis, we generally don't know the exact value of the problem. We have to change error definitions for that purpose.

d) Approximate Error (E_a) : Error calculated from the approximate values of the problem throughout the approximation process .

$$E_a = Current\ Approximation - Previous\ Approximation$$

Similarly with the true error definitions;

e) Approximate Fractional (Relative) Error (ε_a) : It is the fraction of error with respect to approximations.

$$\varepsilon_a = \frac{Current\ Approximation - Previous\ Approximation}{Current\ Approximation}$$

f) Approximate Percent (Relative) Error (%ε_a) :

$$\%\varepsilon_a = \frac{Current\ Approximation - Previous\ Approximation}{Current\ Approximation} x100 = 100\varepsilon_a$$

Example: Maclaurin series: Exponential function.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!}$$

We want to calculate $e^{1.78}$ using Maclaurin series. Starting from the first expression and adding a term at each approximation, calculate $e^{1.78}$. We can select true value of $e^{1.78}$=5.9289856419 as each digit of this number is significant. Calculate error definitions of each approximation. (Use 3 significant digits at error calculations)

1st Iteration:

$e^x \cong 1$ (it contains error, can it be acceptable?)

$$E_t = 5.9289856419 - 1 = 4.93$$
$$\varepsilon_t = \frac{5.9289856419 - 1}{5.9289856419} = 0.831337 = 0.831$$
$$\%\varepsilon_t = \frac{5.9289856419 - 1}{5.9289856419} x100 = 0.831337x100 = 83.1\%$$
$$E_a = X$$
$$\varepsilon_a = X$$
$$\%\varepsilon_a = X$$

2nd Iteration:

$$e^x \cong 1 + x \cong 1 + 1.78 = 2.78$$
$$E_t = 5.9289856419 - 2.78 \quad = 3.15$$
$$\varepsilon_t = \frac{5.9289856419 - 2.78}{5.9289856419} = 0.531117 = 0.531$$
$$\%\varepsilon_t = \frac{5.9289856419 - 2.78}{5.9289856419} x100 = 0.531117x100 = 53.1\%$$
$$E_a = 2.78 - 1 = 1.78$$
$$\varepsilon_a = \frac{2.78 - 1}{2.78} = 0.640288 = 0.640$$
$$\%\varepsilon_a = \frac{2.78-1}{2.78} x100 = 0.640288x100 = 64.0\%$$

3rd Iteration:

$$e^x \cong 1 + x + \frac{x^2}{2!} \cong 1 + 1.78 + \frac{1.78^2}{2!} = 4.3642$$
$$E_t = 5.9289856419 - 4.3642 \quad = 1.57$$
$$\varepsilon_t = \frac{5.9289856419 - 4.3642}{5.9289856419} = 0.263921307 = 0.264$$
$$\%\varepsilon_t = \frac{5.9289856419 - 4.3642}{5.9289856419} x100 = 0.263921307x100 = 26.4\%$$
$$E_a = 4.3642 - 2.78 = 1.58$$
$$\varepsilon_a = \frac{4.3642 - 2.78}{4.3642} = 0.362999 = 0.363$$
$$\%\varepsilon_a = \frac{4.3642-2.78}{4.3642} x100 = 0.362999x100 = 36.3\%$$

4th Iteration:

$$e^x \cong 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$
$$e^{1.78} \cong 1 + 1.78 + \frac{1.78^2}{2!} + \frac{1.78^3}{3!} = 5.3041587$$
$$E_t = 5.9289856419 - 5.3041587 \quad = 0.625$$
$$\varepsilon_t = \frac{5.9289856419 - 5.3041587}{5.9289856419} = 0.105385139 = 0.105$$
$$\%\varepsilon_t = \frac{5.9289856419 - 5.3041587}{5.9289856419} x100 = 0.105385139x100 = 10.5\%$$
$$E_a = 5.3041587 - 4.3642 = 0.940$$
$$\varepsilon_a = \frac{5.3041587 - 4.3642}{5.3041587} = 0.177212 = 0.177$$
$$\%\varepsilon_a = \frac{5.3041587-4.3642}{5.3041587} x100 = 0.177212x100 = 17.7\%$$

5th Iteration:

$$e^x \cong 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$
$$e^{1.78} \cong 1 + 1.78 + \frac{1.78^2}{2!} + \frac{1.78^3}{3!} + \frac{1.78^4}{4!} = 5.7224403$$
$$E_t = 5.9289856419 - 5.7224403 \quad = 0.207$$
$$\varepsilon_t = \frac{5.9289856419 - 5.7224403}{5.9289856419} = 0.0348365439 = 0.0348$$
$$\%\varepsilon_t = \frac{5.9289856419 - 5.7224403}{5.9289856419} x100 = 0.0348365439 x100 = 3.48\%$$
$$E_a = 5.7224403 - 5.3041587 = 0.418$$
$$\varepsilon_a = \frac{5.7224403 - 5.3041587}{5.7224403} = 0.073095 = 0.0731$$
$$\%\varepsilon_a = \frac{5.7224403-5.3041587}{5.7224403} x100 = 0.073095 x100 = 7.31\%$$

6th Iteration:

$$e^x \cong 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$
$$e^{1.78} \cong 1 + 1.78 + \frac{1.78^2}{2!} + \frac{1.78^3}{3!} + \frac{1.78^4}{4!} + \frac{1.78^5}{5!} = 5.8713485$$
$$E_t = 5.9289856419 - 5.8713485 = 0.0576$$
$$\varepsilon_t = \frac{5.9289856419 - 5.8713485}{5.9289856419} = 0.0097212441 = 0.00972$$
$$\%\varepsilon_t = \frac{5.9289856419 - 5.8713485}{5.9289856419} x100 = 0.0097212441 x100 = 0.972\%$$
$$E_a = 5.8713485 - 5.7224403 = 0.149$$
$$\varepsilon_a = \frac{5.8713485 - 5.7224403}{5.8713485} = 0.02536185 = 0.0254$$
$$\%\varepsilon_a = \frac{5.8713485-5.7224403}{5.8713485} x100 = 0.02536185 x100 = 2.54\%$$

7th Iteration:

$e^{1.78}$ ≅5.9155246……………………where to stop!

1.5 Tollerance (ε_s):

It represents the range of error that can be acceptable for the result. When comparing with the tolerance, absolute error is considered.

$|E_t| \leq \varepsilon_s$ stop

or

$|\varepsilon_a| \leq \varepsilon_s$ stop

Which error should we use?

*** In numerical analysis used in engineering, we can not avoid 2 sources of error.

i) Truncation error: We can calculate all the terms of the algorithm, we have to stop at some point. Similarly, we can not show all the digits of the number, we have to stop digit at some point.

ii) Round-off error: We deliberately change the last digit of the number by looking at the following digit. The significant value is not represented correctly but the error involving the number is minimized. However, at each new calculation with the number, the results deviate from the actual value.

Example: The results is : 398.2608317347328783426909210783467331231253465…….. goes on.

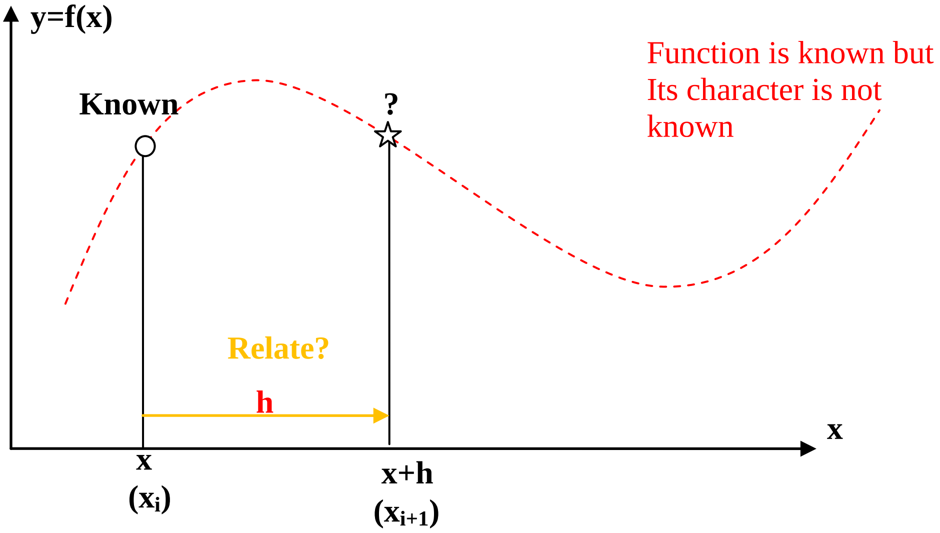
Truncation : we have to cut the digits.
398.260831 /cut/ ..734732878342690921078346
Result is: 398.260831 (truncated, all significant)
Round-off :
398.260831 /cut/ ..7/decide/34732878342690921078346
Result is: 398.260832 (round-off digit, last digit is not significant but error is reduced)

1.6 Taylor Series:

Consider a function which is both continuous and differentiable. (if not?)

Purpose: We want to find the function value using the properties of the function at a different point.

We want to relate the properties of the function at a given point such that we find the function value at another point.



Taylor series expanded at a point to relate the function value at another point is given as:

$$f(x+h) = \underbrace{f(x)}_{0^{th}\text{ order}} + \underbrace{h.f'(x)}_{1^{st}\text{ order}} + \underbrace{\frac{h^2}{2!}f''(x)}_{2^{nd}\text{ order}} + \underbrace{\frac{h^3}{3!}f'''(x)}_{3^{rd}\text{ order}} + \cdots + \underbrace{\frac{h^n}{n!}f^{(n)}(x)}_{n^{th}\text{ order}}$$

Or it is sometimes better to show indices as:

$$f(x_{i+1}) = f(x_i) + h.f'(x_i) + \frac{h^2}{2!}f''(x_i) + \frac{h^3}{3!}f'''(x_i) + \cdots + \frac{h^n}{n!}f^{(n)}(x_i) \quad \text{where}$$

$$h = x_{i+1} - x_i$$

Continuous, differentiable, progress

In numerical analysis, convergence of taylor series is dependent on “h”. If h-->0, series is convergent. If h>1, the series may diverge and this is also dependent on the derivative values.

Taylor series with error term added:

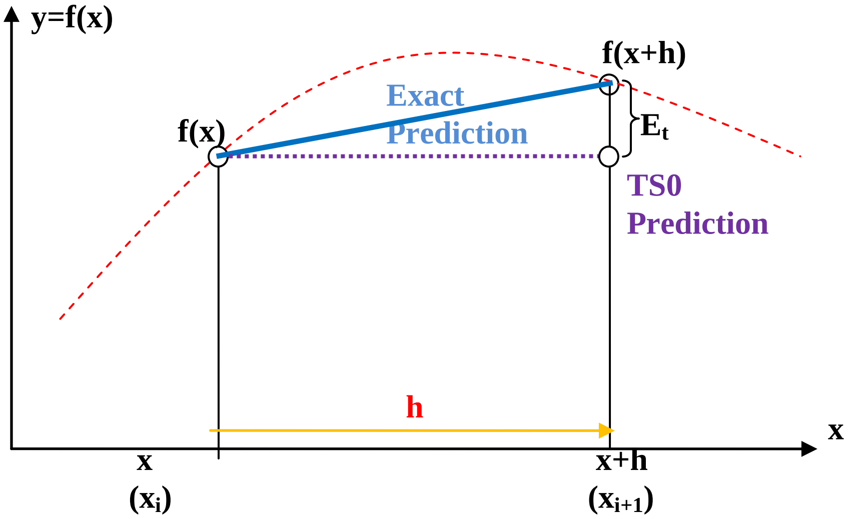
$$f(x+h)=f(x)+\ h.f'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''^{(x)}+\cdots+\frac{h^n}{n!}f^{(n)}(x)\Big|_{truncated}$$
$$+\frac{h^{n+1}}{(n+1)!}f^{(n+1)}(\xi)$$

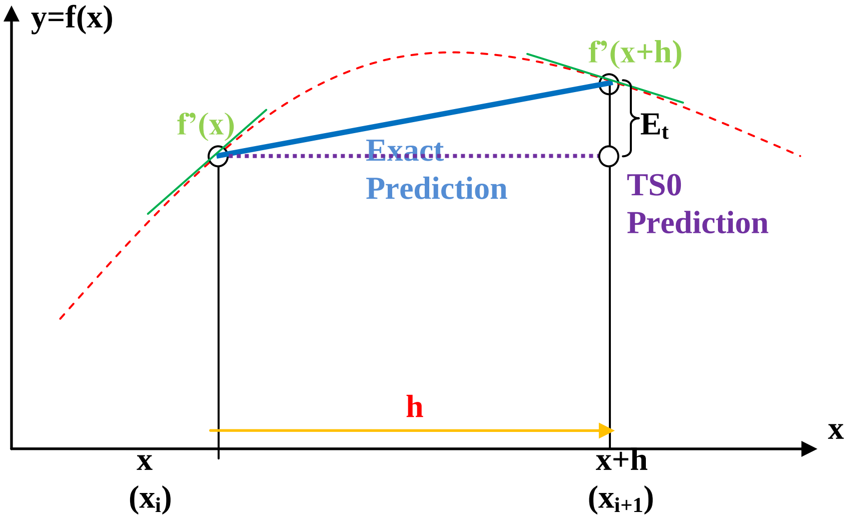
Here, true error defined as:

$$R_n=E_t=\frac{h^{n+1}}{(n+1)!}f^{(n+1)}(\xi)\qquad \text{where}\quad x\leq\xi\leq x+h$$

This term can be calculated from “mean value theorem”. Consider the TS0 expansion series with:

$$f(x+h)\cong f(x)\qquad \text{,which contains } E_t$$

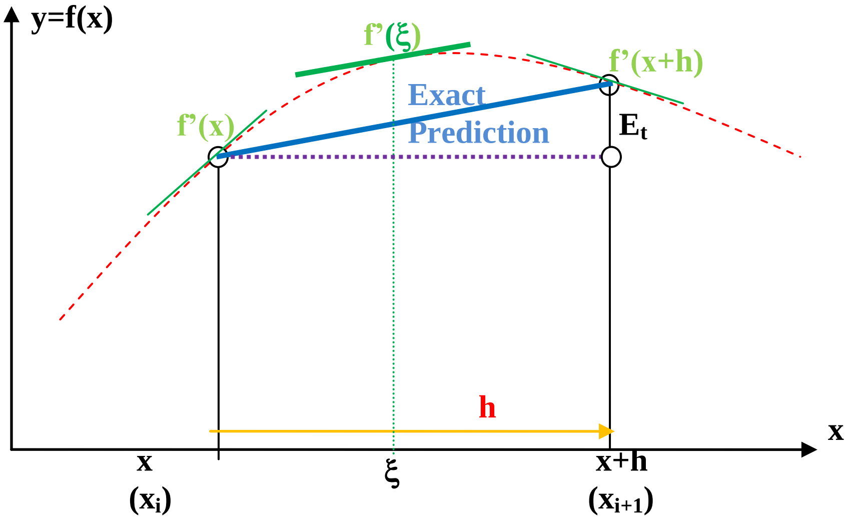




Comparing exact prediction with derivative values at boundaries:

f’(x) ≥ Exact prediction
f’(x+h) ≤ Exact prediction

As function is continuous, the derivative values are continuously changing between f’(x) and f’(x+h) inside the interval. There exists a point inside the interval where the derivative value at that point is equal to exact prediction.



The derivative values of the exact prediction can be written as:

Slope of exact prediction = **f’(ξ) = E$_t$ /h**

Therefore true error of TS0 expansion can be written as:

$$\boldsymbol{E_t = hf'}(\xi) \quad \text{(the exact error of TS0)}$$

Higher orders?

Example: We want to find function value at point x=3 of the following function using Taylor series expansion at point x=2. Starting from 0th order of expansion, calculate approximate value of f(3) until true error satisfies the tolerance of ε_s=0.1. Use 5 significant digits when necessary.

$$f(x) = x^{2.7} - e^{0.5x}$$

Taylor series Expansion:
x=2 , x+h=3 h=(x+h)-x= 3-2=1

$$f(x+h) = f(x) + h.f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots + \frac{h^n}{n!}f^{(n)}(x)$$

Or for this problem;

$$f(3) = f(2) + h.f'(2) + \frac{h^2}{2!}f''(2) + \frac{h^3}{3!}f'''(2) + \cdots + \frac{h^n}{n!}f^{(n)}(2)$$

Luckily, we can find exact value of the problem easily by:

$$f(3) = 3^{2.7} - e^{1.5} = 14.93733445$$

1st iteration (TS0):

$$f(3) \cong f(2)$$

$$f(2) = 2^{2.7} - e^{1} = 3.7797$$

Et=14.93733445 – 3.7797 = 11.158

$$|11.158| \le 0.01 \quad X \quad Continue$$

2^{nd} iteration (TS1):

$$f(3) \cong f(2) + \ h.f'(2)$$

$$f'^{(x)} = \frac{d}{dx}(x^{2.7} - e^{0.5x}) = 2.7x^{1.7} - 0.5e^{0.5x}$$

$$f'(2) = 2.7(2^{1.7}) - 0.5e^{1} = 7.41318$$

$$f(3) \cong 3.7797 + \ 1(7.41318) = 11.193$$

Et=14.93733445 – 11.193 = 3.7444

$$|3.7444| \le 0.01 \quad X \quad Continue$$

3^{rd} iteration (TS2):

$$f(3) \cong f(2) + \ h.f'(2) + \frac{h^2}{2!}f''(2)$$

$$f''^{(x)} = \frac{d}{dx}(2.7x^{1.7} - 0.5e^{0.5x}) = 4.59x^{0.7} - 0.25e^{0.5x}$$

$$f'(2) = 4.59(2^{0.7}) - 0.25e^{1} = 6.7769$$

$$f(3) \cong 11.193 + \frac{1^2}{2}(6.77698) = 14.581$$

Et=14.93733445 – 14.581 = 0.35596

$$|0.35596| \le 0.01 \quad X \quad Continue$$

4^{th} iteration (TS3):

$$f(3) \cong f(2) + \ h.f'(2) + \frac{h^2}{2!}f''(2) + \frac{h^3}{3!}f'''(2)$$

$$f'''^{(x)} = \frac{d}{dx}(4.59x^{0.7} - 0.25e^{0.5x}) = 3.213x^{-0.3} - 0.125e^{0.5x}$$

$$f'(2) = 3.213(2^{-0.3}) - 0.125e^{1} = 2.26998$$

$$f(3) \cong 14.581 + \frac{1^3}{6}(2.26998) = 14.960$$

Et=14.93733445 – 14.960 = -0.022371

$$|-0.022371| \leq 0.1 \quad \surd \ \mathit{Stop}$$

What should we write for the result?

Example: The integral expression:

$$\int_1^2 xe^x dx$$

Can be written as a function as:

$$f(x) = \int_1^x te^t dt$$

and f(x=2) gives the value of the first integral. Using Taylor series expansion of the f(x) function expanded at x=1, calculate the f(2) value with each Taylor Series term added in each iteration. Continue your iteration until true percent error satisfy the tolerance of ε_s=5.

Solution:

For this particular problem, exact solution of the integral can be found as:

$$\int xe^x dx = e^x(x-1)$$

$$\int_1^2 xe^x dx = e^2(2-1) - e^1(1-1) = e^2 = 7.389056099 \ldots..$$

7.389 have accuracy well inside the tolerance. (even 7.4 is enough).

Consider you don’t know the mathematical theorem for integration and can’t calculate the value. Taylor series expansion is one way to solve this problem.

$$f(x) = \int_1^x te^t dt$$

$$f(x+h) = f(x) + \ h.f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots + \frac{h^n}{n!}f^{(n)}(x)$$

x=1 , x+h=2 and h=1. For this problem:

$$f(2) = f(1) + \ h.f'(1) + \frac{h^2}{2!}f''(1) + \frac{h^3}{3!}f'''(1) + \cdots + \frac{h^n}{n!}f^{(n)}(1)$$

1st İteration (TS0)

$$f(2) \cong f(1)$$

$$f(1) = f(x = 1) = \int_{1}^{x=1} te^t dt = \int_{1}^{1} te^t dt = 0$$

$$f(2) \cong 0$$

$$\%\varepsilon_t = \frac{7.389 - 0}{7.389} x100 = \%100$$

$$|100| \leq 5 \quad X \quad Continue$$

2nd Iteration (TS1)

$$f(2) \cong f(1) + \ h.f'(1)$$

$$f'(x) = \frac{d}{dx}\int_{1}^{x} te^t dt = \frac{d}{dx}\int te^t dt\Big|_x - \frac{d}{dx}\int te^t dt\Big|_1$$

$$f'(x) \cong \frac{d}{dx}\int te^t dt\Big|_x = xe^x$$

$$f'(1) \cong 1e^1 = 2.7183$$

$$f(2) \cong 0 + \ 1.(2.7183) = 2.718$$

$$\%\varepsilon_t = \frac{7.389 - 2.718}{7.389} x100 = \%63.2$$

$$|63.2| \leq 5 \quad X \quad Continue$$

3rd Iteration (TS2)

$$f(2) \cong f(1) + \ h.f'(1) + \frac{h^2}{2!} f''(1)$$

$$f''(x) = \frac{d}{dx}(xe^x) = e^x + xe^x$$

$$f''(1) \cong e^1 + 1e^1 = 2e^1 = 5.437$$

$$f(2) \cong 2.718 + \frac{1^2}{2!}(5.437) = 5.437$$

$$\%\varepsilon_t = \frac{7.389 - 5.437}{7.389} x100 = \%26.4$$

$$|26.4| \leq 5 \quad X \quad Continue$$

4th Iteration (TS3)

$$f(2) \cong f(1) + \ h.f'(1) + \frac{h^2}{2!} f''(1) + \frac{h^3}{3!} f'''(1)$$

$$f'''(x) = \frac{d}{dx}(e^x + xe^x) = e^x + e^x + xe^x$$

$$f'''(1) \cong e^1 + e^1 + 1e^1 = 3e^1 = 8.155$$

$$f(2) \cong 5.437 + \frac{1^2}{3!}(8.155) = 6.796$$

$$\%\varepsilon_t = \frac{7.389 - 6.796}{7.389} x100 = \%8.03$$
$$|8.03| \leq 5 \quad X \quad Continue$$

5th Iteration (TS4)

$$f(2) \cong f(1) + \ h.f'(1) + \frac{h^2}{2!} f''(1) + \frac{h^3}{3!} f'''^{(1)} + \frac{h^4}{4!} f^{iv}(1)$$

$$f^{iv}(x) = \frac{d}{dx}(e^x + e^x + xe^x) = e^x + e^x + e^x + xe^x$$
$$f'{}'^{(1)} \cong e^1 + e^1 + e^1 + 1e^1 = 4e^1 = 10.87$$
$$f(2) \cong 6.796 + \frac{1^2}{4!}(10.87) = 7.248$$

$$\%\varepsilon_t = \frac{7.389 - 7.248}{7.389} x100 = \%1.90$$
$$|1.90| \leq 5 \quad \surd \quad Stop$$

Result: 7.248 (you are sure of how many digits?)

FYI:
6th : 7.36201329
7th : 7.38466563
8th : 7.38844102
9th : 7.38898036
10th : 7.38904778
11th : 7.38905527